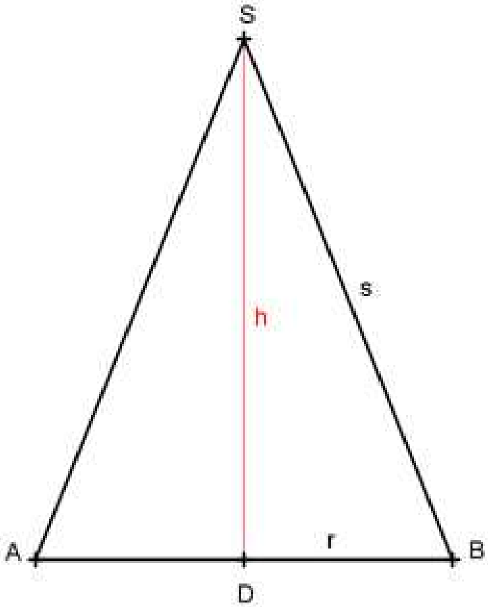
Extrem Aufgabe 123

Wie groß ist die Höhe h eines Kegels, der bei gegebenem Volumen V eine minimale Oberfläche O hat?



Zielfunktion:

$$O = \pi * r^2 + \pi * r * s$$

Nebenbedingung:

$$V = \frac{\pi * r^2 * h}{3}$$

Satz von Pythagoras im Dreieck DBS:

$$s^2 = r^2 + h^2 \quad |\sqrt{\ }$$

$$s = \sqrt{r^2 + h^2}$$

$$V = \frac{\pi * r^2 * h}{3} \quad | * \frac{3}{\pi h}$$

$$r^2 = \frac{V * 3}{\pi h} \quad |\sqrt{\ }$$

$$r = \sqrt{\frac{V * 3}{\pi h}}$$

In die Zielfunktion eingesetzt:

$$O = \pi * r^2 + \pi * r * \sqrt{r^2 + h^2}$$

$$O_{(h)} = \pi * \frac{V * 3}{\pi * h} + \pi * \sqrt{\frac{V * 3}{\pi h}} * \sqrt{\frac{V*3}{\pi h} + h^2}$$

$$O_{(h)} = \frac{V * 3}{h} + \pi * \sqrt{\frac{V^2*9}{\pi^2 h^2} + \frac{V*3}{\pi h} * h^2}$$

$$O_{(h)} = \frac{V * 3}{h} + \pi * \sqrt{\frac{V^2*9+V*3*\pi*h^3}{\pi^2 h^2}}$$

$$O_{(h)} = \frac{V * 3}{h} + \frac{\pi}{\pi h} * \sqrt{9V^2 + 3V\pi h^3}$$

$$O_{(h)} = \frac{3V + \sqrt{9V^2 + 3V\pi h^3}}{h}$$

$0 < h < \infty$
wenn $h \to \infty$, dann $r \to 0$

Quotienten- und Kettenregel:

$$u' = \frac{\frac{1}{2} * 3V * \pi * 3h^2}{\sqrt{9V^2 + 3V\pi h^3}} = \frac{4{,}5 * V * \pi * h^2}{\sqrt{9V^2 + 3V\pi h^3}}$$

$$v' = 1$$

$$O'_{(h)} = \frac{4{,}5 * V * \pi * h^2}{\sqrt{9V^2 + 3V\pi h^3}} * h - 1 * (3V + \sqrt{9V^2 + 3V\pi h^3}) \mid *\sqrt{9V^2 + 3V\pi h^3}$$

$$O'_{(h)} = \frac{4{,}5V\pi h^3 - 3V * \sqrt{9V^2 + 3V\pi h^3} - (9V^2 + 3V\pi h^3)}{\sqrt{9V^2 + 3V\pi h^3}}$$

$$O'_{(h)} = \frac{1{,}5V\pi h^3 - 3V * \sqrt{9V^2 + 3V\pi h^3} - 9V^2}{\sqrt{9V^2 + 3V\pi h^3}}$$

$$\frac{1{,}5V\pi h^3 - 3V * \sqrt{9V^2 + 3V\pi h^3} - 9V^2}{\sqrt{9V^2 + 3V\pi h^3}} = 0 \quad |*\sqrt{9V^2 + 3V\pi h^3}$$

$$1{,}5V\pi h^3 - 3V * \sqrt{9V^2 + 3V\pi h^3} - 9V^2 = 0 \quad | + 3V * \sqrt{9V^2 + 3V\pi h^3}$$

$$1{,}5V\pi h^3 - 9V^2 = 3V * \sqrt{9V^2 + 3V\pi h^3} \quad |^2$$

$$2{,}25V^2\pi^2h^6 - 27V^3\pi h^3 + 81V^4 = 9V^2 * (9V^2 + 3V\pi h^3)$$

$$2{,}25V^2\pi^2h^6 - 27V^3\pi h^3 + 81V^4 = 81V^4 + 27V^3\pi h^3 \quad |-81V^4$$

$$2{,}25V^2\pi^2h^6 - 27V^3\pi h^3 = 27V^3\pi h^3 \quad |+27V^3\pi h^3$$

$$2{,}25V^2\pi^2h^6 = 54V^3\pi h^3 \quad |:V^2\pi h^3$$

$$2{,}25\pi h^3 = 54V \quad | :2{,}25\pi$$

$$h^3 = \frac{24V}{\pi} \quad |\sqrt[3]{}$$

$$\mathbf{h = 2 * \sqrt[3]{\frac{3V}{\pi}}}$$

Zur Beurteilung, ob $O''_{(h)}$ > oder < 0: (Begründung siehe Kurvendiskussion Aufgabe 105)

$$u' = 4{,}5V\pi h^2 - \frac{3V * \frac{1}{2} * 9V\pi h^2}{\sqrt{9V^2 + 3V\pi h^3}} = 4{,}5V\pi h^2 - \frac{3V * 4{,}5V\pi h^2}{\sqrt{9V^2 + 3V\pi h^3}}$$

$$u' = 4{,}5V\pi h^2 * \left(1 - \frac{3V}{\sqrt{9V^2 + 3V\pi h^3}}\right) > 0 \text{ weil } \sqrt{9V^2 + 3V\pi h^3} > 3V$$

$$O''_{(h)} = \frac{u'}{\sqrt{9V^2 + 3V\pi h^3}} = \frac{> 0}{> 0} = > 0 \text{ --> Minimum}$$

$$O_{(2 * \sqrt[3]{\frac{3V}{\pi}})} = \frac{3 * V + \sqrt{9V^2 + 3V\pi * (2 * \sqrt[3]{\frac{3V}{\pi}})^3}}{2 * \sqrt[3]{\frac{3V}{\pi}}}$$

$$O_{(2 * \sqrt[3]{\frac{3V}{\pi}})} = \frac{3 * V + \sqrt{9V^2 + 72V^2}}{2 * \sqrt[3]{\frac{3V}{\pi}}} = \frac{12V}{2 * \sqrt[3]{\frac{3V}{\pi}}} \text{ absolutes Minimum, weil}$$

$$O_{(0)} = \frac{3V + \sqrt{9V^2 + 3V\pi 0^3}}{0} \text{ --> } \infty > \frac{12V}{2 * \sqrt[3]{\frac{3V}{\pi}}}$$

$$O_{(\infty)} = \frac{3V + \sqrt{9V^2 + 3V\pi \infty^3}}{\infty} \text{ --> } \infty > \frac{12V}{2 * \sqrt[3]{\frac{3V}{\pi}}}$$